Propagation of Spatial Soliton in Gaussian Waveguide with Nonlocal Nonlinearity

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Abstract: Propagation of spatial soliton in a waveguide with Gaussian linear refractive index profile (Gaussian waveguide) and nonlocal nonlinearity is investigated. It is shown analytically by equivalent particle approach that in a Gaussian waveguide with local nonlinearity, the soliton oscillates in transverse direction periodically and symmetrically. The present of nonlocal nonlinearity disturbs the soliton oscillation. For weak nonlocal nonlinearity, the soliton still oscillates but asymmetrically. Furthermore the waveguide shows an asymmetric behavior in the sense that a soliton initially launched at a position deviating to the left or to the right side of the waveguide center causes different propagation properties. Stronger nonlocal nonlinearity leads to an exiting soliton from the waveguide. In other words the soliton does not show an oscillatory behavior. The dynamics of soliton are simulated numerically using an implicit Crank-Nicolson scheme and good agreements between our analytical and numerical studies are achieved.

Key words: Spatial soliton, Gaussian waveguide, nonlocal nonlinearity, equivalent particle approach, crank-nicolson scheme.

1. Introduction

During the past decades, solitons have been the object of extensive theoretical and experimental studies [1-4] because of their potential applications, e.g., all optical switching [5, 6], logic devices [7-9], etc. To explore all possible applications, it is very important to understand the generic properties of spatial solitons.

It is well known that the soliton in a homogeneous medium with local Kerr nonlinearity behaves as a particle moving with constant velocity. Introducing a transversal inhomogeneous refractive index distribution in the nonlinear medium may disturb the soliton transversal velocity. Garzia et al. [10] and Ebnali-Heidari et al. [11] have shown the oscillatory behavior of soliton propagating in a waveguide with a Gaussian refractive index profile. In general such behavior is observed when a soliton placed in a symmetrical waveguide [12-15].

Most of above mentioned soliton studies are directed to soliton propagation in medium with local nonlinearity. On the other hand the response of nonlinear medium might be significantly nonlocal which may importantly affect the properties of soliton [16]. For example, a medium with positive nonlocal nonlinearity may induce phenomenon of soliton self-bending during its propagation [17-19]. In a medium with nonlocal nonlinearity, the nonlinear response depends not only on the light intensity but also on its derivative.

The aim of this paper is to study the dynamics of spatial soliton in a Gaussian waveguide with nonlocal nonlinear response. For this purpose, section 2 describes the mathematical model of spatial soliton propagating in inhomogeneous medium with nonlocal nonlinearity. Based on this model, we analyze the soliton dynamics theoretically using equivalent particle approach [20]. To confirm the results of equivalent particle approach, we show and discuss the results of numerical simulations obtained by implicit...
Crank-Nicolson method [21] in section 3. Finally conclusions are presented in section 4.

2. Mathematical Model

The mathematical model describing the dynamics of optical beam in a nonlinear waveguide can be derived from the Maxwell equation. Based on paraxial the approximation and neglecting the anisotropic properties of the medium, the propagation of optical waves along the z axis in a slab waveguide with inhomogeneous linear refractive index in a transversal direction and first-order nonlocal contribution to nonlinear response is modeled by the modified nonlinear Schrödinger equation (m-NLSE) [14]:

\[ i \frac{\partial u}{\partial z} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + |u|^2 u = V u \]  

where \( V = -\Delta n(x) + \mu \frac{\partial |u|^2}{\partial x} \). Here \( \mu \) is the dimensionless slowly varying amplitude envelope, the longitudinal \( z \) and transversal \( x \) coordinates are respectively scaled to the diffraction length and input beam width. The parameter \( \mu \) describes the strength of the nonlocal component of the nonlinear response. To be consistent with the fact that in practice the nonlocal contribution is small compared to the local one, we assume that the parameter \( \mu \) is small. The function \( \Delta n(x) \) in Eq. (1) stands for profile of refractive index distribution. Throughout this paper we consider a waveguide for which the index distribution has a Gaussian profile (Gaussian waveguide):

\[ \Delta n(x) = \Delta n_0 \exp \left(-bx^2\right) \]  

where \( \Delta n_0 \) and \( b \) correspond to the maximum linear refractive index variation and the width of the waveguide, respectively. Notice that the waveguide center is located at \( x = 0 \).

The dynamic of a spatial soliton in a Gaussian waveguide with nonlocal nonlinearity will be studied analytically using equivalent particle approach [20]. First the soliton is considered as a particle whose position is given by \( \bar{x}(z) \) where \( z \) is treated as the time variable. Here we assume that the intensity \( |u|^2 \) moves as a single unit and is a function of \( x - \bar{x}(z) \).

To find an equation for the motion of particle (beam) we then define the beam center as

\[ \bar{x}(z) = p^{-1} \int_{-\infty}^{\infty} x |u|^2 \, dx, \]  

where \( p = \int_{-\infty}^{\infty} |u|^2 \, dx \) is the beam energy (power). From Eq. (1), it follows that the beam power is a conserved quantity and the propagation of the soliton center satisfies the following equations:

\[ v(z) = \frac{d}{dz} \bar{x}(z) = \frac{1}{2p} \int_{-\infty}^{\infty} u^* \frac{\partial u}{\partial x} - u \frac{\partial u^*}{\partial x} \, dx \]  

\[ a(z) = \frac{d^2}{dz^2} \bar{x}(z) = \frac{dV}{dx} \int_{-\infty}^{\infty} |u|^2 \, dx \]  

Because \( p \) is constant, in the mechanical particle analogy, Eq. (4) can be written as Newtonian’s like second law [20]:

\[ \frac{d^2 \bar{x}(z)}{dz^2} + \frac{\partial U(\bar{x})}{\partial \bar{x}} = 0 \]  

where \( U \) is the integral of the right hand side of Eq. (4) with respect to \( \bar{x} \).

We now consider as input a single soliton

\[ u(x,0) = q \sech \left( q(x - \bar{x}_0) \right) \]  

where \( q \) and \( \bar{x}_0 \) are respectively the amplitude and the initial position of soliton. Since we assume that the refractive index perturbations due to the transversal index variation and the nonlocal response are small, we therefore apply a quasi-homogeneous approximation and the perturbed soliton becomes [12]

\[ u(x,z) = q \sech(q(x - \bar{x}(z))) \exp(i(v(z)x + \sigma(z))) \]  

where \( v(z) = d\bar{x}(z)/dz \) and \( d\sigma(z)/dz = \frac{q^2 - v(z)^2}{2} \).

By substituting Eq. (7) into Eq. (4) and Eq. (5), we evaluate the Newton’s potential \( U(\bar{x}) \) and the transversal acceleration \( d\bar{x}(z)/dz \) numerically. Figs. 1-3 show the acceleration and potential energy of the particle-like soliton with zero initial velocity propagating in a Gaussian waveguide with \( b = 1/9 \) for
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Fig. 1  (a) Acceleration and (b) potential of a soliton of amplitude $q = 1$ versus transverse coordinate in a waveguide with $\Delta n_0 = 0.1$, $b = 1/9$ and different values of $\mu$.

Fig. 2  (a) Acceleration and (b) potential of a soliton with different amplitude ($q$) versus transverse coordinate in a waveguide with $\mu = 0.01$, $\Delta n_0 = 0.1$, and $b = 1/9$.

Fig. 3  (a) Acceleration and (b) potential of a soliton of amplitude $q = 1$ versus transverse coordinate in a waveguide with $\mu = 0.01$, $b = 1/9$ and different value of $\Delta n_0$. 
different waveguide parameters ($\Delta n_0$ and $\mu$) and different soliton amplitude $q$. Without nonlocality (Fig. 1a), the acceleration of equivalent particle is an odd function of $\vec{x}$; positive on the left side and negative on the other side. Hence, if a soliton beam is placed at a position shifting either to the left or to the right side of the waveguide center, it will subject to an effective force and then oscillates periodically inside the waveguide. Here the soliton moves back and forth from $\vec{x}_0$ to $-\vec{x}_0$ where $\vec{x}_0$ is the initial position; see also the Newton’s potential in Fig. 1b. The oscillation is symmetric because the acceleration is antisymmetric function. This results agree with Ref. [10].

The present of nonlocality shifts up the acceleration profile and may destroy the potential profile (Figs. 1-3). For the case of weak nonlocality, parts of the acceleration profile remains negative and therefore the Newton’s potential still support the soliton oscillation. In this case the soliton may oscillate periodically but not symmetrically around the waveguide center. Indeed if its initial position is at the right side of waveguide center (say at $\vec{x}_0 > 0$) where the potential still supports the oscillatory behavior, then the back and forth movement will be from $\vec{x}_1$ to $-\vec{x}_2$ with $\vec{x}_2 < \vec{x}_1$. On the other hand, if the initial position is at the left part of the waveguide, i.e., at $-\vec{x}_1$, then the soliton may exit from the waveguide. This asymmetric oscillation is more clearly seen for larger nonlocality. Furthermore, when the nonlocal effects dominate the soliton dynamics then acceleration profile is positive everywhere and the Newton’s potential does not support the soliton oscillation anymore. Here the soliton with any initial position always experience positive acceleration and therefore it will move to the right and exit from the waveguide. The larger nonlocal effects can of course be achieved by increasing the strength of nonlocality ($\mu$) (Fig. 1). However, Fig. 2 shows that in the medium with the same $\mu > 0$, larger nonlocal effects can also be obtained by increasing the soliton amplitude. In Fig. 3, we show the effect of maximum index variation ($\Delta n_0$) on Newton’s acceleration and potential. It can be seen that the higher maximum index variation reduces the domination of nonlocal effects, i.e., the potential well of higher $\Delta n_0$ is deeper than that of lower $\Delta n_0$. This means that a waveguide with higher $\Delta n_0$ will have more possibility to support soliton oscillation.

3. Results of Numerical Simulations and Discussion

To prove the analytical predictions in previous section, we perform some numerical simulations of a spatial soliton propagating in a Gaussian waveguide with nonlocal nonlinearity using implicit Crank-Nicolson scheme [10]. As input beam we take a single soliton Eq. (6). The propagation of soliton with amplitude $q = 1$ in a waveguide with $\Delta n_0 = 0.1$, $b = 1/9$ and different values of $\mu$ are shown in Fig. 4. In the absent of nonlocality ($\mu = 0$), the soliton with initial position slightly deviates to the left or to the right side of the waveguide center oscillates symmetrically inside the waveguide (Figs. 4a and 4b). If we introduce a relatively small nonlocality in the waveguide, i.e., $\mu = 0.01$, the soliton still oscillates in the waveguide (Figs. 4c and 4d). Although the waveguide is structurally symmetric, the waveguide may behave asymmetrically. Figs. 4c and 4d show that the solitons placed at the same distance from the waveguide center may propagate differently. Soliton which is initially placed at the left side of the waveguide ($\vec{x}_0 = -2.5$) has a larger swing distance and larger oscillation period than that at the right side of the waveguide ($\vec{x}_0 = 2.5$). Further observation shows that those oscillations are also not symmetric around the oscillation center. This fact is in accordance with previous theoretical prediction (equivalent particle approach).

The effects of nonlocality are more clearly seen for larger $\mu$ (Figs. 4e and 4-h). In the waveguide with $\mu = 0.025$, the soliton initially launched at $\vec{x}_0 = 2.5$ also oscillates in the waveguide where its period is smaller compared to that in the waveguide with $\mu = 0.01$. On the other hand, in the same waveguide, soliton which
is initially at $x_0 = -2.5$ experiences higher potential energy and therefore it is forced to exit from the waveguide. If we further increase the strength of nonlocality to $\mu = 0.05$ then both solitons with initial positions at $x_0 = \pm 2.5$ exit from the waveguide but with different velocities. All these propagation properties agree with the Newton’s potential shown in Fig. 1b.

Next we perform numerical simulations to study effects of soliton amplitude. In Fig. 5, we show results of numerical simulations of soliton propagating in a Gaussian waveguide with $\mu = 0.01$, $\Delta n_0 = 0.1$ and $b = 1/9$ for three different amplitudes. These numerical results perfectly agree with the predictions of the equivalent particle approach, i.e., a higher amplitude soliton experiences larger nonlocality. Here the soliton with amplitude $q = 0.5$ oscillates in the waveguide because the effect of linear refractive index variation is much stronger than that of nonlocality. The soliton with amplitude $q = 1.0$ experiences higher nonlocal effects than that with $q = 0.5$. In this case the soliton remains oscillating inside the waveguide but the oscillation center is more deviated from the waveguide center and its period is larger. If the soliton amplitude is $q = 1.25$ then the nonlocal effects are much stronger.

In fact the soliton is always pushed to move to the right and exits from the waveguide. This soliton movement is easily explained by the Newton’s potential in Fig. 2b. Indeed the soliton with amplitude $q = 1.25$ and initial position at $x_0 = -2.5$ has a bigger potential energy than the potential energy threshold which supports the soliton oscillation.

To understand the role of maximum index variation $\Delta n_0$, we also perform some simulations of soliton with amplitude $q = 1.0$ propagating in waveguide with $\mu = 0.01$ and $b = 1/9$ by varying $\Delta n_0$. 

**Fig. 4** Propagation of soliton with $q = 1$ in a Gaussian waveguide with $\Delta n_0 = 0.1$, $b = 1/9$, and different values of $\mu$. The initial position is at $x_0 = -2.5$ for the left figures and at $x_0 = 2.5$ for the right figures.
Fig. 5  Soliton propagation in a Gaussian waveguide with \( \mu = 0.01 \), \( \Delta n_0 = 0.1 \) and \( b = 1/9 \) for three different amplitudes.

Fig. 6  Propagation of soliton with amplitude \( q = 1 \) in a Gaussian waveguide with \( \mu = 0.01 \) and \( b = 1/9 \) for three different \( \Delta n_0 \).

The influence of \( \Delta n_0 \) on the dynamics of soliton is clearly seen in Fig. 6. In a waveguide with \( \Delta n_0 = 0.05 \), the soliton escapes from the waveguide since the refractive index perturbation caused by nonlocality is much stronger than by Gaussian linear refractive index distribution. By increasing the maximum index variation to be \( \Delta n_0 = 0.1 \), we observe that the soliton shows an oscillation behavior (Figs. 6b and 4c) for more detail. However, the oscillation is asymmetric and its center deviates from the center of waveguide. In Fig. 6c we show that if the maximum index variation is further increased then the soliton oscillation is getting more symmetric. This fact shows that the effect of linear index distribution is stronger. This phenomenon agrees very well with the prediction of equivalent-particle approach.

4. Conclusions

The dynamics of spatial soliton in a Gaussian waveguide with nonlocal nonlinearity have been investigated analytically using equivalent particle approach and numerically using implicit Crank-Nicolson scheme. Without nonlocality the soliton oscillates symmetrically around the waveguide center. We also show that when the nonlocality is relatively small then the soliton oscillates asymmetrically with the center of oscillation deviates from the waveguide center. Furthermore the waveguide behaves asymmetrically in the sense that a soliton placed in the left or in the right side of the
waveguide center, even with the same distance, leads to different propagation properties. If the nonlocal effect is much stronger than the effect of linear refractive index distribution then the soliton is forced to exit from the waveguide. Stronger nonlocal nonlinearity can be achieved by increasing the strength of nonlocality or the soliton amplitude. On the other hand the domination of nonlocal effect can be reduced by increasing the maximum linear refractive index variation.

References